Probability

SET07106 Mathematics for Software Engineering

School of Computing
Edinburgh Napier University
Module Leader: Uta Priss

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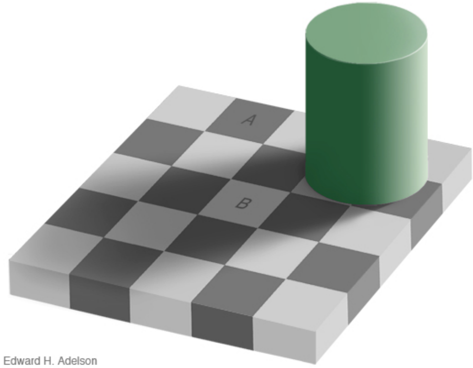
An example

Probability and intuition

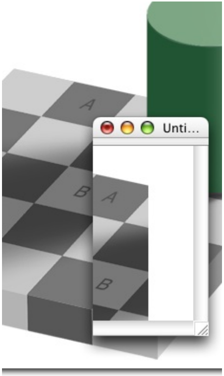
Mathematical probability

Testing

Optical illusion



Optical illusion: evidence

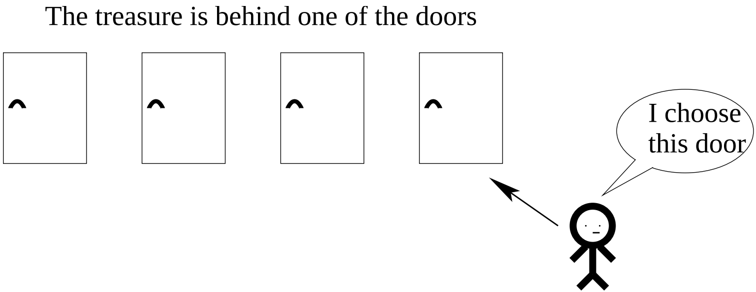


The Monty Hall's game show problem

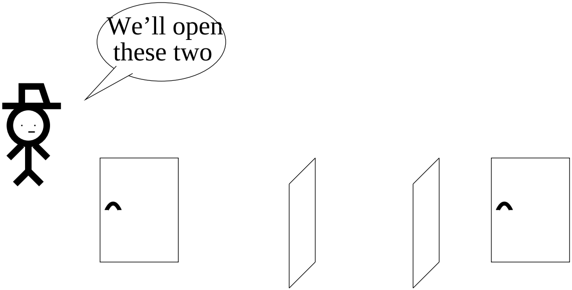
A prize is hidden behind one of 4 doors. A contestant chooses one door (which is NOT opened). The game show host then opens two doors which don't contain the prize. Two doors remain closed. The contestant can now choose whether the originally selected door shall be opened or the other closed door.

What should the contestant do?

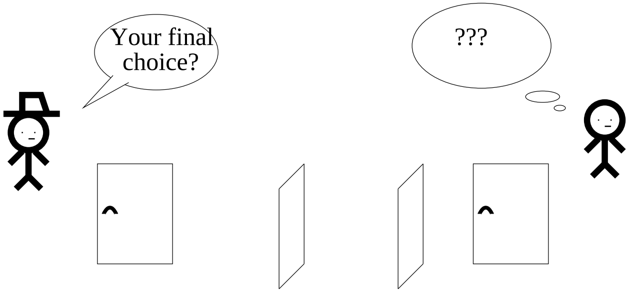
Contestant selects a door (which stays closed)



Two empty doors opened by the host



Contestant makes final choice



Stick with the original choice or select the other closed door?

Intuition about probability

If a prize is behind one of two doors. The probability for each door to contain the prize is 50%.

If a prize is behind one of four doors. The probability for each door to contain the prize is 25%.

Monty Hall's game show: solution

The door which the contestant chose originally contains the prize with a probability of 25%.

Opening two more doors does not change the probability of the original door. It stays 25%.

That means the only other remaining door contains the prize with a probability of 75%.

The contestant should always choose not to have the original door opened, but instead the other door in order to have a winning chance of 75%.

Innumeracy

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(Innumeracy is the mathematical equivalent to illiteracy. John Allen Paulos (1988))

Intuitions about probability

- Practice improves performance.
  Example: shooting an arrow at a target many times will improve during the learning phase.
- Central tendency: if an activity is repeated many times, the result should approach an average.
  Examples: the sequel of a best-selling movie, will be more average; children of unusually tall/short parents tend to be of more average height.

Intuitions about probability (continued)

- Co-occurrence and correlation: if two events co-occur, there is likely to be a connection (assumed causality).
  Example: if someone feels ill after drinking too much, it is likely to be caused by the alcohol.
- Using a random experiment for decision making.
  Examples: "head or tails"; "drawing the short straw"
- An explanation is more likely if it fits a mental model.
  Example: a person who has a university degree is more likely to have a well-paid job than a person without a degree.

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- Correlation: countries with higher meat consumption have higher cancer rates. Does eating meat cause cancer? Or do richer countries have a higher life expectancy?
  (There can be a hidden cause, or co-occurring events might have no relationship at all.)

Pigeon hole principle

If 366 people are in the same room, at least two will have the same birthday. Why?

Example: what is the probability that 2 people have different birthdays?

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Probability that 2 people have different birthdays:

Calculating the probability of an event: $\frac{\text{all relevant events}}{\text{all possible events}}$

Probability that 2 people have different birthdays: $\frac{365 \times 364}{365 \times 365}$

Probability that 2 people have the same birthday:

$$1 - \frac{\text{all relevant events}}{\text{all possible events}} = 1 - \frac{365 \times 364}{365 \times 365} = \frac{365}{365} - \frac{364}{365} = \frac{1}{365}$$

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Three people: $1 - \frac{\text{all relevant events}}{\text{all possible events}} = 1 - \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = 0.008$

Keep trying with 4, 5, 6, until value > 0.50.

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23 people are needed.

(This is a lot less than the 366 people that are needed to have 2 people with the same birthday with 100% certainty.)

Rules about probability

- Probability is a value between 0 and 1 (or 0% and 100%).
- An impossible event has probability 0.
- A certain event has probability 1.
- Complement: $P(\text{not } A) = 1 - P(A)$
- Two events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Two independent events: $P(A \cap B) = P(A) \times P(B)$

Probability of two events

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$P(A) \geq P(A) \times P(B)$

Throwing dice

Probability of throwing at least one 1 with two dice:

Throwing one 1 with one dice:

Throwing dice

Probability of throwing at least one 1 with two dice:

Throwing one 1 with one dice: $\frac{1}{6}$.
Throwing two 1s with two dice:

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Throwing one 1 with one dice: $\frac{1}{6}$.
Throwing two 1s with two dice: $\frac{1}{36}$.
Throwing at least one 1 with two dice:
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

11 possibilities:
(1,1),(2,1),(1,2),(3,1),(1,3),(4,1),(1,4),(5,1),(1,5),(6,1),(1,6)

What is the probability of getting at least 1 “head” with 2 coins?

Winning the lottery

Which combination of numbers is more probable:
1, 2, 3, 4, 5, 6 or 13, 25, 27, 35, 41, 45?

Winning the lottery

Which combination of numbers is more probable:
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All possibilities to draw 6 out of 49: $\frac{49!}{6! \times (49-6)!} = \frac{1 \times 2 \times 3 \times \ldots \times 49}{(1 \times 2 \times \ldots \times 6) \times (1 \times 2 \times \ldots 43)} = \frac{44 \times 45 \times 46 \times \ldots \times 49}{1 \times 2 \times \ldots \times 6} = 13,983,816$

Probability of getting all 6 right: $\frac{1}{13{,}983{,}816}$

Inferential statistics

Using probability theory to

- estimate a value (for example a mean)
- test a hypothesis
- make a prediction

Testing

A test for cancer correctly diagnoses cancer in 98% of the tests. In 2% of the tests, the result is false positive, i.e. the test shows cancer even though the patient is healthy.

Assumptions:
10,000 people are tested with this cancer test and the cancer is fairly rare (only 0.5 % of people have this cancer).

If a patient is told that the test is positive, should the patient worry?

Testing (continued)

Test is 98% correct:
$98/100 \times 0.5/100 \times 10{,}000 = 49$

Test has 2% false positives:
$2/100 \times (1 - 0.5/100) \times 10{,}000 = 199$

$\Rightarrow$ Test has 248 positive results.
$49/248 \approx 20\%$ probability to have cancer if test positive.