Boolean or Propositional Logic

SET07106 Mathematics for Software Engineering

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2010

Outline

Boolean logic

Implication

Defining a set

Propositional Logic

Boolean logic

Consider variables $A, B, C, ...$ which can have two values: either True (1) or False (0).

There are three logical operators: and, or, not.

Negation: $\text{not}(\text{not A}) = \text{A} \quad \text{not}(1) = 0 \quad \text{not}(0) = 1$

Complements: $\text{A or (not A)} = 1 \quad \text{A and (not A)} = 0$

Implication: $\text{A} \implies \text{B}$

Where is this logic used?

In many programming languages, query languages, search engines.
Using different notations:

and: &, &&
or: $|, ||$
not: $\sim$, !

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if ( ... and ... )
if (not ( ... ) or ... )
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Boolean logic properties

- associative:
  (A and B) and C $=$ A and (B and C)
  (A or B) or C $=$ A or (B or C)
- commutative:
  A and B $=$ B and A
  A or B $=$ B or A
- distributive:
  (A and B) or C $=$ (A or C) and (B or C)
  (A or B) and C $=$ (A and C) or (B and C)
- idempotent:
  A and A $=$ A
  A or A $=$ A
- transitive: A $\Longrightarrow$ B and B $\Longrightarrow$ C implies A $\Longrightarrow$ C

Logic versus natural language

The logical use of "and, or, not" can be quite different from how these are used in natural language.

Formal logic can be counter-intuitive.

AND

What does "and" mean in these sentences:

- He entered the room and sat down.

AND

What does "and" mean in these sentences:

- He entered the room and sat down. $\Longrightarrow$ then
- She bought a computer and a printer.

AND

What does “and” mean in these sentences:

- He entered the room and sat down. $\Longrightarrow$ then
- She bought a computer and a printer. $\Longrightarrow$ and
- Students in classes 101 and 202.

AND

What does “and” mean in these sentences:

- He entered the room and sat down. $\Longrightarrow$ then
- She bought a computer and a printer. $\Longrightarrow$ and
- Students in classes 101 and 202. $\Longrightarrow$or

OR

What does “or” mean in these sentences:

- Would you like a beer or a whisky.

OR

What does "or" mean in these sentences:

- Would you like a beer or a whisky.
  $\Longrightarrow$ **exclusive or**: "either or" (BOTH would be impolite)
- I bet he is sitting in the bar and drinking a beer or a whisky.

OR

What does "or" mean in these sentences:

- Would you like a beer or a whisky.
  $\Longrightarrow$ **exclusive or**: "either or" (BOTH would be impolite)
- I bet he is sitting in the bar and drinking a beer or a whisky.
  $\Longrightarrow$ **inclusive or**: (BOTH is acceptable)

Logical "or" is always **inclusive**: "one or the other or both".

NOT

- Rhetoric uses: The drink was not bad.

NOT

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- Double negative: I doN'T DISlike computers.

NOT

- Rhetoric uses: The drink was not bad.
- Double negative: I doN'T DISlike computers. $\implies$ positive
- Double negative: We doN'T need NO education.

NOT

- Rhetoric uses: The drink was not bad.
- Double negative: I doN'T DISlike computers. $\Longrightarrow$ positive
- Double negative: We doN'T need NO education. $\Longrightarrow$ negative

Logical "not not A" always means "A".

Truth Tables

not(name = 'Smith' or age = '40')

name	age	name or age	not(name or age)
true	true	true	false
true	false	true	false
false	true	true	false
false	false	false	true

De Morgan's law

- not (a and b) = (not a) or (not b)
- not (a or b) = (not a) and (not b)

He doesn't want tea or coffee.
He doesn't want tea and he doesn't want coffee.

Exercise

Use de Morgan's laws to show that the complement of

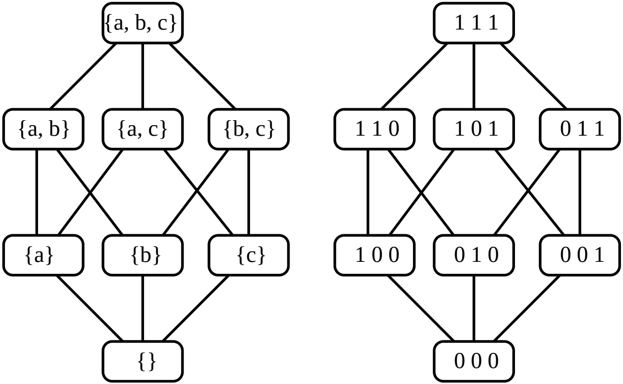
$(\overline{A}$ and $B)$ and $(A$ or $\overline{B})$ and $(A$ or $C)$

is

$(A$ or $\overline{B})$ or $(\overline{A}$ and $(B$ or $\overline{C}))$

Note: $\overline{A}$ means: not A.

Set theory and Boolean logic



Implication

$A \implies B$ means (not A) or B

A	B	not(A)	(not A) or B
true	true	false	true
true	false	false	false
false	true	true	true
false	false	true	true

True implies true.
True can never imply false.
If A is false, then anything can be implied.

Examples of implication: $A \Longrightarrow B$

- A is true:
  - If $x = 2$, then $2x = 4$.
  - If you practice for an exam, then you will succeed.
- A is false:
  - If I were a carpenter, then I would be rich.
  - If $5 = 7$, then $15 = 22$.

Exercise: is this true or false?

If Sue is a programmer, then she is smart.
If Sue is an early riser, then she does not like porridge.
If Sue is smart, then she is an early riser.

Therefore, if Sue is a programmer, then she does not like porridge.

Contraposition

A $\implies$ B implies not B $\implies$ not A.

Example:

If it is raining, I carry an umbrella.
If I don't carry an umbrella, it is not raining.

The following are false:

If it is not raining, I do not carry an umbrella.
If I carry an umbrella, it is raining.

Exercise

Prove that contraposition is true, i.e.
A $\implies$ B implies not B $\implies$ not A.

Hint: you can use truth tables or the properties of Boolean logic for this.

Extensional and intensional definition of sets

- $\star$ Extensional:

$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$\{2, 4, 6, 8, 10\}$
$\{2, 3, 5, 7, 11, ...\}$

- $\star$ Intensional:

$\{n \mid 1 \leq n \leq 10\}$
$\{m \mid m = 2n \text{ and } 1 \leq n \leq 5\}$

$\{n \mid n \text{ is a prime number }\}$
$= \{n \mid \forall_k : k \text{ is a factor of } n \implies n = 1 \text{ or } n = k\}$

Intensional definition of sets

$\{n \mid \forall_k : k \text{ is a factor of } n \implies n = 1 \text{ or } n = k\}$

⋆ name of the resulting variable: n

⋆ definition: $\forall_k : k \text{ is a factor of } n \implies n = 1 \text{ or } n = k$

What symbols can be used in definitions?

Here: $\forall_k$, $=$, "is a factor of", and, or, not

The Liar's Paradox

The Liar's Paradox

Either:
The Liar speaks the truth $\Rightarrow$ The Liar lies.
Or:
The Liar lies $\Rightarrow$ "I am a liar" is wrong. The Liar speaks the truth.

Russell's paradox: Is there a set of all sets?

Intensional: $S := \{T \mid T \text{ is a set }\}$
Extensional: $S := \{$ set of integers, set of real numbers, set of traffic light symbols, set of all things in the universe, ..., S $\}$.

But then one can also define:

$\overline{S}$: the set of all sets that do not contain itself.
But then since $\overline{S}$ does not contain itself, it must contain itself!

What does this mean for Set Theory?

- Sets must be carefully defined.
- Not every collection of elements that can be described in words is necessarily a set.
- In extensional definitions of infinite sets, it must be clear what "..." means.
- In intensional definitions, it must be clear what symbols can be used.

If one isn't sure whether something is a set, call it a "class".
There is no "set of all sets", but there is a "class of sets".

In intensional definitions, it must be clear what symbols can be used:

A **formal logic** is a language with a fixed set of symbols with syntax, grammar, semantics (formal meaning) which can be used for defining sets and for reasoning, deduction and inference.

Exercise

Which of these are sets, which are classes?

- all elements in the universe
- $n \mid n < 5$ and $n > 5$
- all dinosaurs which ever lived on the Earth
- all dinosaurs which would be alive now, if some catastrophe had not killed their species
- all sets
- all subsets of a set

Propositional logic

A proposition is a statement that is either true or false.

Exercise: which of these are propositions?

- $1 + 1 = 2$
- How are you?
- I am fine.
- x $==$ 3
- if (x $==$ 3) $\{$
- print 'Hello World'
- n $=$ n $+$ 1

Formal propositions

Propositions are formed using "and, or, not" and variables (i.e. Boolean Logic).

The semantics (or meaning) of a proposition is its truth value.

Two propositions p, q are equivalent ($\mathrm{p} \iff \mathrm{q}$) if they always have the same truth value. (They are either both true or both false.)

Exercise: which of these are equivalent?

- $1 + 1 = 2$ and $2 + 2 = 4$
- It is raining today.
- 8 is a prime number.
- $x = 3 \Longleftrightarrow x = 4$
- There are 25 students in this classroom.

Intensional definitions of sets

Propositional logic in combination with some mathematical symbols ($=$, $\leq$, $\times$) can be used to define sets, such as:

$\{2, 4, 6, 8, 10\} = \{m \mid m = 2 \times n \text{ and } 1 \leq n \leq 5\}$

But propositional logic is not sufficient for this definition:

$\{n \mid \forall_k : k \text{ is a factor of } n \implies n = 1 \text{ or } n = k\}$

because $\forall_k$ is not part of propositional logic.