Diagrammatic Representation of Conceptual Structures *

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Abstract. Conceptual exploration as provided by Formal Concept Analysis is potentially suited as a tool for developing learning materials for teaching mathematics. But even just a few mathematical notions can lead to complex conceptual structures which may be difficult to be learned and comprehended by students. This paper discusses how the complexity of diagrammatic representations of conceptual structures can potentially be reduced with Semiotic Conceptual Analysis. The notions of "simultaneous polysemy" and "observational advantage" are defined to describe the special kind of relationship between representations and their meanings which frequently occurs with diagrams.

1 Introduction

When learning mathematics, students need to acquire concepts using some kind of informal conceptual exploration where they mentally verify implications and identify counter examples to rule out misconceptions. The Formal Concept Analysis (FCA) method of conceptual exploration can formalise such a process by determining the relevant implicit knowledge that is contained in a domain. But concept lattices tend to contain too many concepts to be individually learned. Most likely some form of covering of the content of a concept lattice is required. Diagrams and visualisations often provide very concise representations of knowledge, because, as the proverb states, "a picture is worth 1000 words". But visualisations have both advantages as well as limits. Different students may be more or less adept in reading graphical representations. Thus not a single, but a variety of forms of representations may be required and students need to learn to switch between them.

A supportive theory for understanding the role of diagrams for conceptual learning is supplied by Semiotic Conceptual Analysis (SCA) – a mathematical formalisation of core semiotic notions that has FCA as a conceptual foundation (Priss 2017). A sign in SCA is a triple consisting of an interpretation, a representamen and a denotation (or meaning) where interpretations are partial functions from the set of representamens into the set of denotations. The capacity of a representamen (such as a diagram) to denote more than just one meaning is introduced in the notion of *simultaneous polysemy* in this paper. Simultaneous polysemy is contrasted with *ambiguous polysemy* which describes a representamen being mapped onto slightly different meanings in different usage contexts. Both forms of polysemy contribute to the efficiency of sets of representamens:

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a smaller set of representamens is capable of representing a larger set of denotations. For example, it is more efficient and easier to learn to read and write languages that use an alphabet as compared to Chinese. Letters of an alphabet are ambiguously polysemous because each letter expresses a small variety of different but similar phonemes in different usage contexts. Diagrams are usually simultaneously polysemous which is further explored in the notion of *observational advantage* adopted from the research about diagrams by Stapleton et al. (2017), although with a somewhat different formalisation.

The next section introduces a few non-standard FCA notions and the core SCA notions required for this paper. An introduction to FCA is not provided because FCA is the main topic of this conference. Section 3 discusses observational advantages of tabular, Euler and Hasse diagrams. Section 4 provides a small example of how to obtain an observationally efficient diagram. The paper finishes with a conclusion.

2 FCA and SCA Notions

This section repeats relevant notions from SCA, introduces some new SCA notions and a few non-standard FCA notions. A *supplemental concept* is a concept whose extension equals the union of the extensions of its proper subconcepts. In a Hasse diagram with minimal labelling, supplemental concepts are those that are not labelled by an object. Each supplemental concept corresponds to a clause because for such a concept c with extension $ext(c)$ and intension $int(c)$ and the condition $\forall(o_i \in ext(c) : \exists(c_i < c : o_i \in ext(c_i)))$ it follows that $\bigwedge(a_i \in int(c)) \Rightarrow \bigvee(a_i \mid \exists c_i : c_i < c, a_i \in int(c_i), a_i \notin int(c))$ is a clause. Supplemental concepts are particularly interesting if the formal context is non-clarified and contains all objects that are known to be possible within a domain. In that case a clause presents not just information about the concept lattice but instead background knowledge about the domain.

In this paper the notion *conceptual class* is used as a placeholder for a formalisation of conceptual structures. A conceptual class is a structure consisting of sets, relations and functions. A *logical description of a conceptual class* $\mathcal{L}(\mathcal{C})$ is defined as a set of true statements according to the rules of some logical language $\mathcal{L}$. A logical description could be provided by a description logic, formal ontology or other formal language. Further details are left open so that SCA can be combined with a variety of conceptual formalisations.

The following definitions briefly summarise SCA. More details can be found in Priss (2017). A sign is a triple or element of a triadic relation:

Definition 1. For a set R (called *representamens*), a set D (called *denotations*) and a set I of partial functions $i : R \nrightarrow D$ (called *interpretations*), a *semiotic relation* S is a relation $S \subseteq I \times R \times D$. A relation instance $(i, r, d) \in S$ with $i(r) = d$ is called a *sign*. For a semiotic relation, an equivalence relation $\approx_R$ on R, an equivalence relation $\approx_I$ on I, and a tolerance relation $\sim_D$ on D are defined.

Additionally, the non-mathematical condition is assumed that signs are actually occurring in some communication event at a certain time and place. Representamens are entities that have a physical existence, for example as a sound wave, a neural brain pattern, a text printed in a book or on a computer screen or a state in a computer. Perceiving a sound wave or a pattern on a computer screen as a word is already an interpretation.

Denotations represent meanings of signs usually in the form of concepts. Because of their physical existence, representamens tend to be at most equivalent instead of equal to each other. For example, two spoken words will never be totally equal but can be considered equivalent if they are sufficiently similar to each other. When referring to a sign, the notions "sign" and "representamen" are sometimes used interchangeably because a representamen is the perceptible part of a sign. For example, one might refer to "four" as a sign, word or representamen.

The tolerance relation in Def. 1 expresses similarity amongst meanings corresponding to synonymy. For example, "car" might be a synonym of "vehicle" and "vehicle" a synonym of "truck", but "truck" not a synonym of "car". In some domains (such as mathematics), equality of denotations is more important than similarity. The tolerance relation also serves the purpose of distinguishing polysemous signs (with similar meanings) from homographs which have totally unrelated meanings. For example (i_1, "lead", [some metal]) and (i_2,"lead", [to conduct]) are homographs whereas the latter is polysemous to (i_3,"lead", [to chair]). An equivalence relation on interpretations can express a shared usage context of signs consisting of time, place and sign user. Equivalent interpretations belong to a single, shared usage context and differ only with respect to some further aspects that do not define a usage context. The first part of the next definition is repeated from Priss (2017) but the rest is new in this paper.

Definition 2. For a semiotic relation S, two signs (i_1, r_1, d_1) and (i_2, r_2, d_2) are *synonyms* $\iff d_1 \sim_D d_2$. They are *polysemous* $\iff r_1 \approx_R r_2$ and $d_1 \sim_D d_2$. Two polysemous signs are *simultaneously polysemous* if $i_1 \approx_I i_2$. Otherwise they are *ambiguously polysemous*.

Formally any sign is polysemous to itself, but in the remainder of this paper an individual sign is only called polysemous if another sign exists to which it is polysemous. Ambiguous polysemy refers to a representamen being used in different usage contexts with different meanings. The usage context disambiguates such polysemy. Ambiguous polysemy poses a problem if the ambiguity cannot be resolved by the usage context. A benefit of ambiguous polysemy is that a small number of representamens can refer to a much larger number of denotations because each representamen can occur in many usage contexts. Ambiguous polysemy involving different sign users expresses differences in understanding between the users. For example a student and a teacher might use the same terminology but not have exactly the same understanding of it.

Definition 3. For a semiotic relation S with a sign $s := (i, r, d)$, the sets $S_{sp}(s) := \{s\} \cup \{(i_1, r_1, d_1) \mid \exists_{s_2 \in S_{sp}(s)} : (i_1, r_1, d_1)$ polysemous to s_2 and $i_1 \approx_I i\}$ and $D_{sp}(s) := \{d_1 \mid \exists (i_1, r_1, d_1) \in S_{sp}(s)\}$ are defined. For $S_1 \subseteq S$ the sets $S_{sp}(S_1) := \bigcup_{s \in S_1} S_{sp}(s)$ and $D_{sp}(S_1) := \bigcup_{s \in S_1} D_{sp}(s)$ are defined.

The semiotic relation $S_{sp}(s)$ contains all signs that have interpretations that can be applied to r within the same usage context. It should be noted that $|S_{sp}(s)| > |D_{sp}(s)|$ is possible because two interpretations of signs in $S_{sp}(s)$ can map r onto the same denotation. For ambiguous polysemy we have in the past suggested to use neighbourhood lattices[1] (Priss & Old 2004). In the terminology of SCA, such neighbourhood lattices only consider a binary relation between representamens and denotations. A neighbour-

[1] Neighbourhood lattices were originally invented by Rudolf Wille in an unpublished manuscript.

hood context is formed by starting with a denotation and finding all representamens that are in relation with it, then all other denotations which are in relation with one of the representamens and so on. Alternatively, one can start with a representamen. Depending on the sizes of the retrieved sets one can determine when to stop and whether to apply different types of restrictions (Priss & Old 2004). Def. 3 suggests a similar approach for simultaneous polysemy: $D_{sp}(s)$ and $D_{sp}(S_1)$ retrieve all denotations belonging to a representamen or set of representamens with equivalent interpretations.

In some cases, a sign s has a representamen r which has parts that are representamens of signs (e.g. s_1) themselves. In that case a mapping from s to s_1 is called an *observation* if an additional non-mathematical condition is fulfilled that the relationship is based on some perceptual algorithm. For a diagram and the set S_1 of all signs that can be simultaneously observed from it, $D_{sp}(S_1)$ models an "observational advantage" in analogy to the notion of Stapleton et al. (2017). In domains such as mathematics, an *implication* can be considered to hold between signs if it is logically valid amongst the denotations of the signs. Observations, however, hold between signs based on representamens. Ideally observations amongst signs should imply implications and, thus, representamen-based relationships should correlate with or at least not disagree with denotation-based relationships.

Definition 4. For a semiotic relation S with signs $s := (i, r, d)$, $s_1 := (i_1, r_1, d_1)$, $i \approx_I i_1$ and $d = d_1$, the sign s has an *observational advantage* over s_1 if $|D_{sp}(s)| > |D_{sp}(s_1)|$. For two semiotic relations S_1 and S_2 whose interpretations all belong to the same equivalence class, S_1 has a *higher observational efficiency* over S_2 if $D_{sp}(S_1) = D_{sp}(S_2)$ and $|S_{sp}(S_1)| < |S_{sp}(S_2)|$. A semiotic relation S has *maximal observational advantage* over a conceptual class if $D_{sp}(S)$ contains the set of true statements of the logical description of the conceptual class.

The last sentence of the definition extends the notion of observations to a relationship between signs and conceptual classes where a conceptual class is purely denotational and not a semiotic relation itself. Alternatively, conceptual classes could also be formalised as semiotic relations. Observational advantage and efficiency can be considered a measure of quality for representamens. A sign with a higher observational advantage might be better for certain purposes because it provides more information. A semiotic relation has a higher observational efficiency if it can express the same content with fewer signs. With respect to semiotic relations, the measure could be further refined, because otherwise just containing a single sign with an observational advantage is sufficient to cause a semiotic relation to have a higher observational efficiency.

Flower, Fish & Howse (2008) distinguish concrete and abstract diagrams. A concrete diagram is actually drawn whereas an abstract diagram contains all the information that is required for producing a concrete diagram. For example, an abstract Hasse diagram describes nodes, edges, labels and their relationships to each other. A concrete diagram also contains x- and y- coordinates, fonts, colours and so. Such distinctions are, for example, relevant for the planarity of graphs: an abstract diagram is called planar if a concrete drawing without line crossings is possible. In SCA, an interpretation maps a concrete diagram onto an abstract diagram (as a denotation) which can then be considered a representamen and mapped onto a conceptual class. In general, it is always a matter of judgement to choose interpretations and denotations of a semiotic relation.

For example, one can argue that a (concrete or abstract) Hasse diagram of a concept lattice has maximal observational advantage over its concept lattice. This follows directly from how Hasse diagrams are defined, but still depends on how a conceptual class of concept lattices is defined. Formal contexts as diagrams might contain fewer representamens than Hasse diagrams of their concept lattices and could thus have a higher observational efficiency. But it can be argued that only a binary relation can be observed from a formal context. Constructing a lattice from a binary relation involves implications as well as observations. While it may be possible to read maximum rectangles from a formal context, most people would have great difficulty observing the complete conceptual ordering and, for example, the top and bottom concepts from a formal context. In any case, both formal context diagrams as well as Hasse diagrams of concept lattices have maximal observational advantage over the binary relation between objects and attributes.

3 Tabular, Euler and Hasse Diagrams

Euler diagrams are a form of graphical representation of subsets of a powerset that is similar to Venn diagrams but leaves off any zones that are known to be empty. Not all subsets of a powerset can be represented by an Euler diagram in a *well-formed* manner without including some supplemental zones (similar to supplemental concepts). Supplemental zones are often shaded in order to indicate that they are empty. Fig. 1 shows an Euler diagram in the middle. It is slightly unusual because its curves are boxes instead of circles or ellipses. The correspondence between the Euler diagram and the Hasse diagram on the right should be evident from the letters. The reason for drawing the Euler diagram with boxes is because it allows a reduction to a diagram shown on the left which is called *tabular diagram* in this paper. It seems that there should be an established notion for "tabular diagrams" but there seem to be a variety of similar notions (mosaic plots/displays, contingency tables, Karnaugh maps) which all have slightly different additional meanings. In a sense, tabular diagrams are a 2-dimensional version of the "linear diagrams" invented by Leibniz (Chapman et al. 2014).

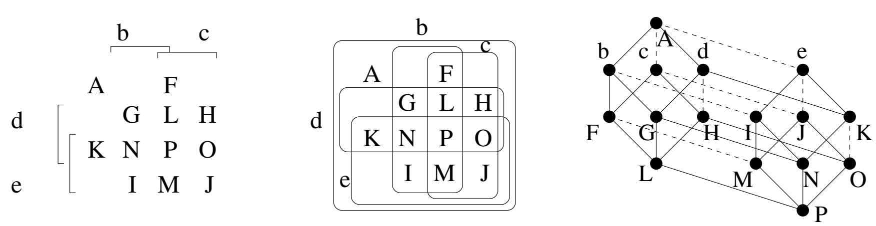

Fig. 1. Tabular, Euler and Hasse diagrams

Considered as single representamens, each diagram in Fig. 1 is simultaneously polysemous because it contains a large amount of information: objects, attributes, their binary relation and implications amongst attributes. The three types of diagrams can be considered to denote the same conceptual class and thus to have the same observational advantage. Instead of considering each diagram as one representamen, it can also be considered as a semiotic relation consisting of parts that are representamens. For the following theorem it is assumed that sets (or curves) with multiple labels are not allowed and that each diagram contains more than one set.

Theorem: Tabular, Euler and Hasse diagrams denote a shared conceptual class corresponding to partial orders of sets with labels and elements. If they exist, tabular diagrams have a higher or equal observational efficiency than Euler diagrams which have a higher observational efficiency than Hasse diagrams.

It should be noted that an Euler diagram might just be a partially ordered set, not a lattice. A translation from Euler to Hasse diagrams is discussed by Priss (2020) and shall be omitted here. The proof of the second half of the theorem is that tabular diagrams contain exactly one representamen for each object, at most two representamens for each attribute (the name of the attribute and a bracket) and nothing else. Euler diagrams also contain one representamen for each attribute and object, and one curve for each attribute. Because a bracket in a tabular diagram can be omitted if the attribute belongs to just one row or column and the outer curve may be omitted, tabular diagrams contain potentially fewer representamens than Euler diagrams. Hasse diagrams contain labels for the objects and attributes, one node for each attribute, but also some edges, thus more representamens than Euler diagrams.

The question arises as to which concept lattices and which Euler diagrams can be represented as tabular diagrams. Any tabular diagram can be converted into an Euler diagram by extending the brackets into boxes but the resulting Euler diagram may not be well-formed. For more than 4 elements, it may not be possible to construct a tabular diagram at all. A solution is to duplicate some of the row and column labels, if the diagram is not possible otherwise. Being able to embed a lattice into a direct product of two planar lattices, is not sufficient as a condition for a corresponding tabular diagram. Petersen's (2010) description of an "S-order" characterises lattices which correspond to 1-dimensional linear diagrams and could lead to a characterisation. Answering such questions may be as difficult as it is to determine which Euler diagrams are well-formed (Flower, Fish & Howse 2008) which has only been solved by providing algorithms so far.

Observability as defined in the previous section is a formal condition. It does not imply that all users can actually observe the information. If a diagram gets too large and complex, users will have difficulties observing anything. Also, some people are more, some less skilled in reading information from graphical representations. Furthermore, we are not suggesting that tabular or Euler diagrams have a higher observational efficiency over Hasse diagrams with respect to all possible conceptual classes nor that they are in any other sense "superior" to lattices. In fact a comparison between Euler and Hasse diagrams shows varied results (Priss 2020).

One of the disadvantages of Euler and tabular diagrams is that some of the structural symmetry may be missing. The dashed lines in Fig. 1 indicate relationships between concepts that are neighbours in the Hasse diagram (and in the ordering relation), but are not neighbours in the tabular diagram. Thus observability has many aspects to it. Modelling with different semiotic relations and conceptual classes will produce different results. Psychological aspects relating to perception exist in addition to formal aspects.

4 Obtaining an Observationally Efficient Diagram

This section employs an example of a formal context and lattice from Ganter & Wille (1999, Section 2.2) consisting of seven prototypical types of triangles and their properties (Fig. 2). The example is discussed using a 3-step investigation: 1. conceptual exploration, 2. reduction of the concept lattice using background knowledge and 3. finding a diagrammatic representation that has a high observational efficiency.

The corresponding lattice is shown in Fig. 2 with empty nodes representing supplemental concepts and filled nodes for non-supplemental concepts. More than half of the concepts are supplemental – and adding more types of triangles as objects would not change that. If one removes the attribute "not equilateral" (and stores it as background knowledge), then the resulting lattice in the left half of Fig. 3 contains fewer supplemental concepts. Since there is only one object that has the attribute "equilateral", this information can still be observed from the lattice. Thus the complexity of the lattice can be reduced if some of the information is stored as background knowledge.

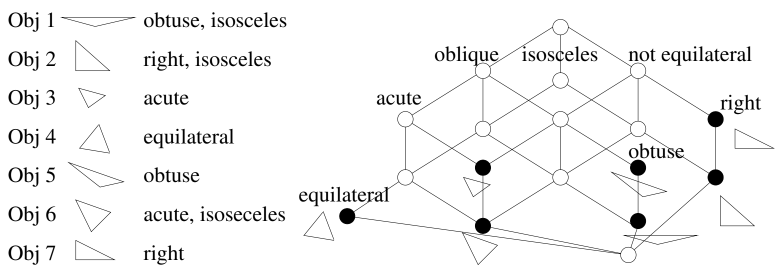

Fig. 2. Classification of triangles (according to Ganter & Wille (1999))

The third step consists of considering observational advantages and efficiency. By representing formal objects as diagrams of triangles, each object visually contains the information about which attributes it has. Thus, for example, the triangle for Obj 4 is simultaneously polysemous because the fact that it is equilateral, acute, oblique and isosceles can be observed from it if one knows what having such an attribute looks like. If one uses a string "equilateral triangle" for Obj 4 then only the attribute "equilateral" can be observed. The other attributes can be inferred but not observed. Representing formal objects as diagrams provides observational advantages over presenting them as strings. Of course, in most applications it will not be the case that objects can be represented as diagrams. Furthermore, in some disciplines diagrams are more suitable than in others.

The tabular diagram on the right of Fig. 3 provides a higher observational efficiency as explained in the previous section. If a student wants to memorise all existing types of

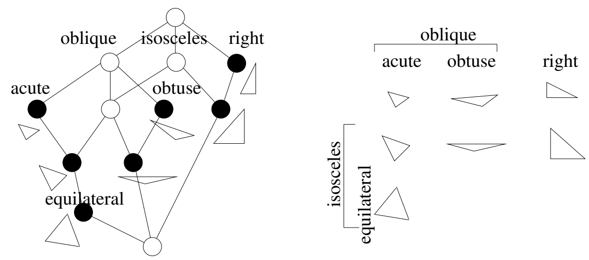

Fig. 3. Classification of triangles: Hasse and tabular diagram

triangles, then maybe this is the most suitable diagram. Apart from the bottom node of the lattice, the other supplemental concepts are still structurally present in the tabular diagram which becomes clear if the curves of the Euler diagram are added. If the attribute "not equilateral" was added then an Euler diagram would also have more empty zones, but those would not be visible in the tabular diagram. Again, we are not suggesting that in general hiding information is an advantage. It depends on the purpose of a diagram.

5 Conclusion

This paper discusses means of obtaining diagrammatic representations of conceptual structures that provide observational advantages and high observational efficiency. A background for this research is to find ways of developing teaching material that covers a topic area efficiently but also, if possible, by connecting to visual structures. Mathematical statements and proofs often present a combination of information that can be observed and information that must be known or inferred. We believe that in particular the notion of "seeing" information in representations is not yet fully understood, even though there is a long tradition of and large body of research in diagrammatic reasoning and information visualisation.

References

1. Chapman, P.; Stapleton, G.; Rodgers, P.; Micallef, L.; Blake, A. (2014). *Visualizing sets: An empirical comparison of diagram types.* In: International Conference on Theory and Application of Diagrams, Springer, p. 146-160.
2. Flower, J.; Fish, A.; & Howse, J. (2008). *Euler diagram generation.* Journal of Visual Languages & Computing, 19, 6, p. 675-694.
3. Ganter, B.; Wille, R. (1999). *Formal Concept Analysis. Mathematical Foundations.* Berlin-Heidelberg-New York: Springer.
4. Petersen, W. (2010). *Linear Coding of Non-linear Hierarchies: Revitalization of an Ancient Classification Method.* In: Advances in Data Analysis, Data Handling and Business Intelligence, Springer, p. 307-316.
5. Priss, U.; Old, L. J. (2004). *Modelling Lexical Databases with Formal Concept Analysis.* Journal of Universal Computer Science, Vol 10, 8, 2004, p. 967-984.
6. Priss, U. (2017). *Semiotic-Conceptual Analysis: A Proposal.* International Journal of General Systems, Vol. 46, 5, p. 569-585.

7. Priss, U. (2020). *Set Visualisations with Euler and Hasse Diagrams.* In: Graph Structures for Knowledge Representation and Reasoning: 6th International Workshop, Springer, LNCS.
8. Stapleton, G., Jamnik, M., & Shimojima, A. (2017). *What makes an effective representation of information: a formal account of observational advantages.* Journal of Logic, Language and Information, 26, 2, p. 143-177.